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Constructing a Perfect Reconstruction System by a Quincunx Sampling Grid

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ABSTRACT

In this study, a matrix grid **D** is used to analyze and synthesize a multidimensional signal comprising an important and useful tool for sublatticed processing in the decomposition and reconstruction of a signal into its polyphase components. Also presented is a filter bank with polyphase components functioning freely as aliases when using a quincunx matrix to specify obligatory and sufficient conditions and to simplify the requisite procedures for determining the number of variables and equations for a perfect reconstruction of an analysis/synthesis system.

***Key Words*: quincunx matrix grid, polyphase filter bank, perfect reconstruction

利用棋盤型取樣網建立之完整回復系統

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摘　要

本文中作者利用棋盤型的取樣原則，敘述多維空間的訊號在經過刪樣及增樣過程，使得多維空間訊號在分解及組合中，能減少所需要的方程式及變數的數目，並且能夠產生完整回復之多頻、多相位濾波庫系統的充要條件。

關鍵詞：棋盤型取樣網，多相位濾波庫，完整回復

I. INTRODUCTION

In signal processing, critically sampled wavelet transforms are known as filter banks or subband transforms [6, 7]. In mathematical analysis, wavelets are defined as translates and dilates of one fixed function and are used to analyze and represent general functions. Multiresolution analysis provides the connection between filter banks and wavelets. An obvious way to build wavelets in a higher dimension is through tensor products of one-dimensional constructions resulting in separable filters. However, this approach gives preferential treatment to the coordinate axes and only allows for rectangular divisions of the frequency spectrum. Often symmetry axes and certain nonrectangular divisions of the frequency spectrum correspond better to the human visual system.

In the past two decades using different lattices (separable or non-separable) as sampling grid to design a filter bank in multidimension has been became widely adopted to the perfect reconstruction system in the applications of signal processing [1-3, 5, 8]. Investigating the relationship between phases coefficients in a two-channel perfect reconstruction filter bank, either set of coefficients is sufficient for perfect reconstruction, and it is of interest to study the problem of phase shifting, i.e., how to obtain odd-phase coefficients from even-phase coefficients or vice versa [2, 5].

The outline of this paper is as follows. In section II we first review some of the element topics that will be used throughout this paper, and then we introduce the relations between separable and non-separable by using different grid matrices, finally we build the z-transform of a signal after downsampling and upsampling, and show the effect. In section III we use the quincunx matrix to specialize the necessary and sufficient condition for perfect reconstruction of an analysis/synthesis system.

II. SUBBAND SCHEMES

At the beginning of this section, we review the basic concepts of multidimensional multirate systems. For signals that are functions of time, the term multirate refers to system in which the sampling rate of signals at different points is different.

The integer lattice $\mathbf{\Lambda}$ is defined as the set of all integer vectors $\mathbf{n}=(n_1, \dots n_N)^T$. With $\mathbf{z}$ defined as the complex vector $\mathbf{z}=(z_1, \dots z_N)^T$, and $\mathbf{z}^{\mathbf{n}}$defined by

$$\mathbf{z}^{\mathbf{n}} = \prod_{i=1}^{N} z_i^{n_i} \tag{1}$$

Definition 1: The N-dimensional z-transform of a sequence $x(\mathbf{n})= x(n_1,\dots n_N)$ is defined as

$$X(z_1,\cdots z_N) = \sum_{n_1}\cdots\sum_{n_N} x(n_1,\cdots n_N) z_1^{-n_1}\cdots z_N^{-n_N} \tag{2}$$

We will write this in the following form:

$$X(\mathbf{z}) = \sum_{\mathbf{n}\in\Lambda} x(\mathbf{n})\mathbf{z}^{-\mathbf{n}} \tag{3}$$

in order for the two expressions for $X(\mathbf{z})$ to be identical, $\mathbf{z}^{\mathbf{n}}$ must be defined as follows.

Definition 2: The operation of raising an N-dimensional complex vector $\mathbf{z}$ to an N-dimensional integer vector $\mathbf{n}$ is defined as

$$\mathbf{z}^{\mathbf{n}} = \prod_{i=1}^{N} z_i^{n_i} \tag{4}$$

We will be manipulating z-transform expressions that involve a sequence and its polyphase components defined on a sampling $\mathbf{D}$. The z-transforms of the polyphase components can be represented in terms of the complex vector $\mathbf{z}^{\mathbf{D}}$, which is defined as follows.

Definition 3: Let d_l denote the lth column of the matrix $\mathbf{D}$. The quantify $\mathbf{z}^{\mathbf{D}}$ is defined as a complex vector $(\mathbf{z}_1,\cdots,\mathbf{z}_N)$ whose lth component $\mathbf{z}_l$ is given by

$$\mathbf{z}_l = \mathbf{z}^{d_l} = \prod_{i=1}^{N} z_i^{d_{il}} \tag{5}$$

$$\mathbf{z}^{\mathbf{D}} = (\mathbf{z}_1,\cdots,\mathbf{z}_N)^{\mathbf{D}} = \left(\mathbf{z}^{d_1},\cdots,\mathbf{z}^{d_N}\right) = \left(\prod_{i=1}^{N} z_i^{d_{i1}},\cdots,\prod_{i=1}^{N} z_i^{d_{iN}}\right) \tag{6}$$

With the definition of $\mathbf{z}^{\mathbf{D}}$, the following property holds.

Property 1:

$$\left(\mathbf{z}^{\mathbf{D}}\right)^{\mathbf{n}} = \mathbf{z}^{\mathbf{Dn}} \tag{7}$$

Property 2: if $\mathbf{z}$ is an N-dimensional complex vector, and $\mathbf{m}=(m_1,\dots m_N)$ is an N-dimensional integer vector, then

$$(-\mathbf{z})^{\mathbf{m}} = (-1)^{\sum_{k=1}^{N} m_k}\ \mathbf{z}^{\mathbf{m}} \tag{8}$$

Property 3: if $\mathbf{I}$ is the $N\times N$ identity matrix and $\mathbf{z}$ is an N-dimensional complex vector, then

$$\left(-\mathbf{z}\right)^{-\mathbf{I}} = -\left(z^{-\mathbf{I}}\right) \tag{9}$$

Property 4: Space Reversal: if $y(\mathbf{n}) = x(-\mathbf{n})$, then $Y(\mathbf{z}) = X\left(\mathbf{z}^{-\mathbf{I}}\right)$.

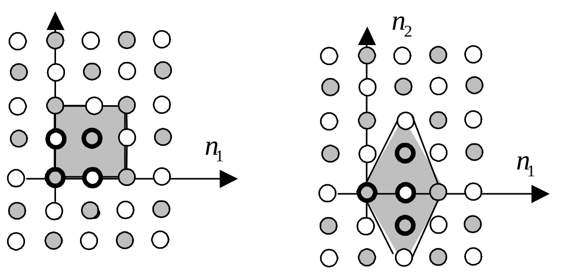
1. Subsampling with Dilation Matrices

Wavelets are known to have a tight relation with subband schemes, and also known as filter banks. In this subsection, we are mainly concerned with filter banks for 2D signals with two bands. The sampling lattice $\mathbf{\Lambda_D}$ generated by the sampling matrix **D** is the set of all integer vector **m** such that **m**=**Dn** for some integer vector **n**. A sampling matrix **D** must be nonsingular with integer value entices. A coset of a sublattice is the set of points obtained by shifting the entire sublattice by an integer shift vector **k**. There are exactly |**det D**| distinct cosets of $\mathbf{\Lambda_D}$, and their union is the integer lattice $\mathbf{\Lambda}$. We refer the vector **k** associated with a certain coset as a coset vector.

A square matrix **D** with integer entries is said to be a *dilation matrix* if the absolute values of its eigenvalues are larger than 1. In subband schemes, dilation matrices are used to specify the subsampling lattice. Every lattice is defined with a corresponding *dilation matrix* **D**. In general, for a *d*-dimensional sampling this matrix will have *d* rows and *d* columns. Columns of matrix **D** are vectors that form a basis of a given lattice.

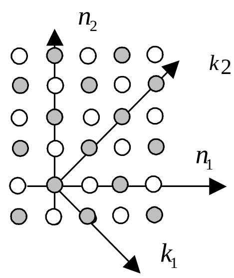
For a two-dimensional case, we need a grid **D** to analyze and syntheses a signal *x*, where **D** is a 2×2 matrix with *M*= |det **D**|; hence there are *M* cosets and *M* samples in the fundamental parallelepiped. For the quincunx matrix, it needs only two channels to reconstruct a two-dimensional signal **x**, so, it is more efficient than the separable case which needs four cosets indeed (see Figure 1 and Figure 2).

There are many ways to subsample an image into a number of subsets. Figure 1 shows a separable lattice and a hexagonal lattice.

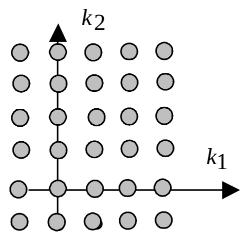


a. Separable b. Nonseparable

Fig. 1.Two different sampling schemes



a. Samples of the original integer lattice Λ



b. First coset in the subsampled domain

Fig. 2. Samples are renumbered in such a way that the overall effect is their 45° counterclockwise rotation

For the 2-channel case, the first task is to split the input image into 2 equally large distinct sets of pixels. This is a more sophisticated task than that in the 1-D case (downsampling or upsampling by 2). For the quincunx dilation matrix $\mathbf{D}=\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ has determinant *M*=2 which splits the set $\mathbf{Z}^2$ into two cosets, the first coset with grey circle and the second with white circle in Figure 2. We use the way which is similar to the downsampling by two in one-dimensional. The first coset is derived by

$$\mathbf{DZ^2=\{m|m=Dn,\ n\in Z^2\}} \tag{10}$$

Where $\mathbf{m}=(m_1,\ m_2)^T$, $\mathbf{n}=(n_1,\ n_2)^T$, and the second is $\mathbf{DZ^2+u}$, where $\mathbf{u}=(1,0)^T$ which is the shift of the first coset by $(1, 0)^T$, shown in Figure 2.

2. The z-transform of a Nonseparable Quincunx Sampling

An important and useful tool in multirate processing is the decomposition of a signal into its polyphase components. Let $\mathbf{m_0},\ldots,\mathbf{m}_{M\text{-}1}$ be a set of coset vectors associated with sampling matrix **D**, and *M*= |**det D**|. The *l*th polyphase component $x_l(\mathbf{n})$ of a signal $x(\mathbf{n})$ is formed by shifting $x(\mathbf{n})$ by $-\mathbf{m}_l$ and by downsampling the result:

$$x_l(\mathbf{n}) = x(\mathbf{Dn}+\mathbf{m}_l) \tag{11}$$

A signal can be recovered from its polyphase components simply by upsampling, appropriately shifting, and summing them. Let $\mathbf{m} = \mathbf{Dn}+\mathbf{m}_l$ then $x(\mathbf{m}) = x_l(\mathbf{n})$, and

$$X(\mathbf{z}) = \sum_{\mathbf{m}\in\Lambda} x(\mathbf{m})\mathbf{z}^{-\mathbf{m}} = \sum_{l=0}^{M-1}\sum_{\mathbf{n}\in\Lambda} x_l(\mathbf{n})\mathbf{z}^{-(\mathbf{Dn}+\mathbf{m}_l)}$$
$$= \sum_{l=0}^{M-1} X_l\left(\mathbf{z}^{\mathbf{D}}\right)\mathbf{z}^{-\mathbf{m}_l} \qquad (12)$$

The first identity states the z-transform of a signal $x(\mathbf{n})$, while the last identity states the polyphase components of the z-transform of a signal $x(\mathbf{n})$ Fig. 3.

Example 1: For a two-dimensional signal $x(\mathbf{n})\in l_2(\mathbf{Z}^2)$, $\mathbf{n}\in\mathbf{Z}^2$ can be factorized into polyphase components $x_l(\mathbf{n})$, l=0,1 related with the cosets $\Lambda_\mathbf{D}$, and $\Lambda_\mathbf{D}+\mathbf{m}_1$ where $\mathbf{m}_1=(1,\ 0)^T$ and the quincunx dilation matrix $\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}$, the results:

$$x(\mathbf{n}) = x_0(\mathbf{n}) + x_1(\mathbf{n}) = x(\mathbf{Dn}) + x(\mathbf{Dn}+\mathbf{m}_1).$$

and the z-transform of $x(\mathbf{n})$

$$X(\mathbf{z}) = X_0\left(z_1 z_2^{-1},\ z_1 z_2\right) + z^{-1}X_1\left(z_1 z_2^{-1},\ z_1 z_2\right)$$

The downsampler samples its input $x(\mathbf{n})$ by mappimg points on the sublattice $\Lambda_\mathbf{D}$ to Λ according to

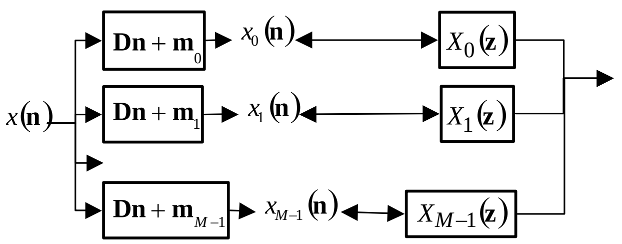


Fig. 3. The *M*-channel ployphases components of signal

$$y(\mathbf{n}) = x(\mathbf{Dn}) \qquad (13)$$

and by discarding samples of $x(\mathbf{n})$ not on $\Lambda_\mathbf{D}$, shown in Figure 4. The samples on horizontal axes of $y(\mathbf{n})$ are transformed from the samples of signal $x(\mathbf{n})$ on the line

$$\begin{cases} n_2 = n_1 + 2k_1 \\ n_2 = -n_1 + 2k_2 \end{cases} \quad k_1, k_2 \in \mathbf{Z}\,. \qquad (14)$$

If we use the sampling matrix **D**, the signal y, which is the downsampling of x with **D**, is denoted by: $y=\downarrow\mathbf{D}(x)$; that is, $y(\mathbf{n}) = x(\mathbf{Dn})$ are the samples of signal at the first coset. Then the z-transform $Y(\mathbf{z})$ of y is

$$Y(\mathbf{z}) = \sum_{\mathbf{n}\in\Lambda} y(\mathbf{n})\mathbf{z}^{-\mathbf{n}} = \frac{1}{M}\sum_{l=0}^{M-1} X\left[\mathbf{e}_{\mathbf{D}^{-1}}(2\pi\mathbf{k}_l)\mathbf{z}^{\mathbf{D}^{-1}}\right] \qquad (15)$$

Where $\mathbf{e}_\mathbf{D}(\boldsymbol{\omega}) = \left(e^{-i\boldsymbol{\omega}^T d_0},\ \cdots,\ e^{-i\boldsymbol{\omega}^T d_{M-1}}\right)^T$, and d_l is the *lth* column of **D**.

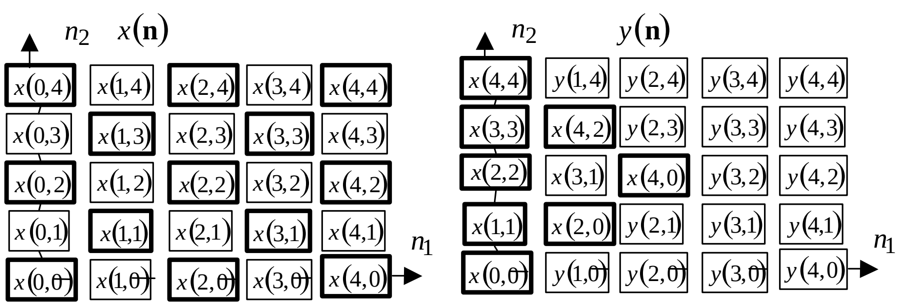
While the output y now is the upsampling of x with matrix **D**, and denoted by $y=\uparrow\mathbf{D}(x)$, that is,

$$y(\mathbf{n}) = \begin{cases} x\left(\mathbf{D}^{-1}\mathbf{n}\right) & \text{if } \mathbf{n}\in\Lambda_\mathbf{D} \\ 0 & \text{otherwise} \end{cases} \qquad (16)$$

And the z-transform of y is:

$$Y(z) = \sum_{\mathbf{n}\in\mathbf{Z}^2} y(\mathbf{n})\mathbf{z}^{-\mathbf{n}} = \sum_{\mathbf{n}\in\Lambda_\mathbf{D}} x\left(\mathbf{D}^{-1}n\right)\mathbf{z}^{-\mathbf{n}} = \sum_{\mathbf{n}\in\mathbf{Z}^2} x(\mathbf{n})\mathbf{z}^{-\mathbf{Dn}} = X\left(z^{\mathbf{D}}\right) \qquad (17)$$

Example 2: If we use the quincunx matrix $\mathbf{D}=\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}$, with



a. Those with bold rectangular of signal $x(\mathbf{n})$ belong to the first coset $\Lambda_\mathbf{D}$

b. The downsampling of the first coset of $x(\mathbf{n})$

Fig. 4. Components and downsampling of a 5×5 signal by the matrix $\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}$

transpose $\mathbf{D}^T=\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}$, associated with the shift vectors $\mathbf{k}_0 = (0,0)^T$ and $\mathbf{k}_1=(0,1)^T$ in the parallelepiped generated by columns of $\mathbf{D}^T$. The output y, $y=\downarrow\mathbf{D}(x)$, then the z- transform $Y(z)$ of y is

$$Y(\mathbf{z})=\sum_{\mathbf{n}\in\Lambda} y(\mathbf{n})\mathbf{z}^{-\mathbf{n}}=\frac{1}{2}\left(X\left[\mathbf{z}^{\mathbf{D}^{-1}}\right]+\mathbf{z}^{-1}X\left[-\mathbf{z}^{\mathbf{D}^{-1}}\right]\right)$$
$$=\frac{1}{2}\left(X\left[\left(z_1^{1/2}z_2^{1/2},\quad z_1^{-1/2}z_2^{1/2}\right)\right]\right.$$
$$\left.+\,z^{-1}X\left[-z_1^{1/2}z_2^{1/2},\ -z_1^{-1/2}z_2^{1/2}\right]\right) \quad (18)$$

The output y after upsampling x with dilation matrix $\mathbf{D}$ is $\uparrow\mathbf{D}(x)$, that is

$$y(\mathbf{n})=\begin{cases}x\left(\mathbf{D}^{-1}\mathbf{n}\right) & \text{if } \mathbf{n}\in\Lambda_{\mathbf{D}}\\ 0 & \text{otherwise}\end{cases} \quad (19)$$

$$Y(\mathbf{z})=\sum_{\mathbf{n}\in\mathbf{Z}^2} y(\mathbf{n})\mathbf{z}^{-\mathbf{n}}=X\left(z_1 z_2^{-1}, z_1 z_2\right) \quad (20)$$

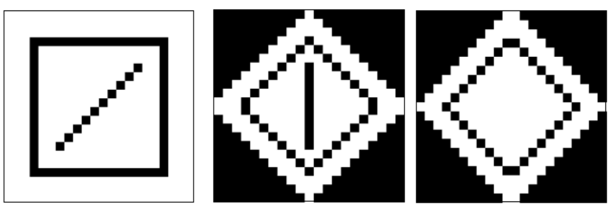
3. The Effeteness of the Quincunx Sampling

A quincunx polyphase decomposition of one simple image is shown in Figure 5. Notice the effect of the counterclockwise rotation by 45 degree. It is interesting to note that the one pixel wide diagonal line is completely transferred in the first phase while it completely disappears in the second phase.

III. A MODULATION AND REVERSAL FILTER BANKS AND PERFECT RECONSTRUCTION

1. Filter and Perfect Reconstruction of an Analysis and Synthesis System

A multidimensional uniform M -channel analysis/ synthesis which is shown in Figure 6 maximally sampled filter bank, all channels share the same sampling matrix grid $\mathbf{D}$, and



(a) (b) (c)

Fig. 5. Quincunx polyphase decomposition: the original image (a) is split in two phases (b) and (c)

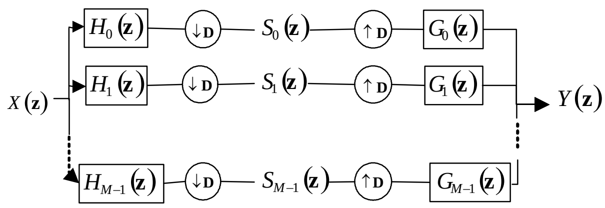


Fig. 6. An uniform M-channel analysis/synthesis system

the number of channels equals M. In a typical application, the subband component vector $\mathbf{S}(\mathbf{z})$ is processed in some fashion prior to passing through the synthesis bank.

For an M-channel analysis/synthesis system,

$$Y(\mathbf{z})=\frac{1}{M}\left[G_0(\mathbf{z}),\cdots,G_{M-1}(\mathbf{z})\right]=\mathbf{H}_{AC}(\mathbf{z})\begin{bmatrix}X\left[e_{\mathbf{D}^{-1}}(2\pi\mathbf{k}_0)\mathbf{z}\right]\\ X\left[e_{\mathbf{D}^{-1}}(2\pi\mathbf{k}_1)\mathbf{z}\right]\\ \vdots\\ X\left[e_{\mathbf{D}^{-1}}(2\pi\mathbf{k}_{M-1})\mathbf{z}\right]\end{bmatrix} \quad (21)$$

Where $\mathbf{H}_{AC}(\mathbf{z})$ is an $M\times M$ matrix with the (m,l)th element is $H_m\left[e_{\mathbf{D}^{-1}}(2\pi\mathbf{k}_l)\mathbf{z}\right]$, which is simply a modulated or aliased version on $H_m(\mathbf{z})$. For this reason $\mathbf{H}_{AC}(\mathbf{z})$ is known as the aliasing component matrix. The $\mathbf{k}_l$'s are the coset vectors for $\mathbf{D}^T$, with $\mathbf{k}_0 = (0,\ldots 0)^T$. In this case, $e_{\mathbf{D}^{-1}}(2\pi\mathbf{k}_0)=1$, and the first element of the right-most vector $X\left[e_{\mathbf{D}^{-1}}(2\pi\mathbf{k}_0)\mathbf{z}\right]$ is $X(\mathbf{z})$. The remaining elements $X\left[e_{\mathbf{D}^{-1}}(2\pi\mathbf{k}_l)\mathbf{z}\right]$, for $l = 1,\ldots,M\text{-}1$ are aliased versions of $X(\mathbf{z})$.

It is evident that the filter bank output is identical to the input if and only if

$$\begin{bmatrix}G_0(\mathbf{z})\\ G_1(\mathbf{z})\\ \vdots\\ G_{M\text{-}1}(\mathbf{z})\end{bmatrix}=M\,\mathbf{H}_{AC}^{-T}(\mathbf{z})\begin{bmatrix}1\\ 0\\ \vdots\\ 0\end{bmatrix} \quad (22)$$

Thus for a filter bank to achieve PR, it is necessary and sufficient that the AC matrix be invertible and the synthesis filters be chosen as the first column of $\mathbf{H}_{AC}^{-T}(\mathbf{z})$.

The polyphase form of a filter bank is obtained by decomposing both the analysis and synthesis filters into their respective polyphase components and by moving the polyphase components through the downsamplers and upsamplers, which is shown in the Figure 7. Let $\mathbf{m}_0,\ldots,\mathbf{m}_{M\text{-}1}$ be a set of polyphase shift vectors for $\mathbf{D}$ and define the analysis and the synthesis polyphase components by the equations:

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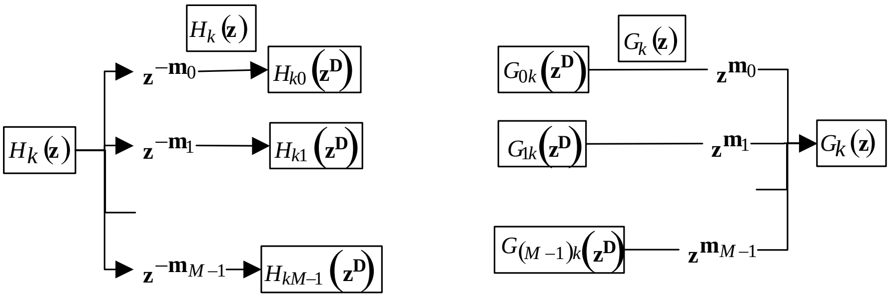

Fig. 7. (Left) The *k*th component of analysis system, and (right) the *k*th component of synthesis system

$$H_k(\mathbf{z}) = \sum_{l=0}^{M-1} \mathbf{z}^{-\mathbf{m}_l} H_{kl}\left(\mathbf{z}^{\mathbf{D}}\right) \quad (23)$$

$$G_k(\mathbf{z}) = \sum_{l=0}^{M-1} \mathbf{z}^{\mathbf{m}_l} G_{lk}\left(\mathbf{z}^{\mathbf{D}}\right) \quad (24)$$

Note that the polyphase forms of analysis and synthesis banks are not defined in exactly the same way. The matrices $\mathbf{H}_P(z)$ and $\mathbf{G}_P(z)$ are known as the analysis and the synthesis polyphase matrices, respectively. We see that the filter bank achieves PR if

$$\mathbf{G}_P(\mathbf{z})\mathbf{H}_P(\mathbf{z}) = \mathrm{I} \quad (25)$$

Because in that case, the analysis bank simply decomposes the input into its polyphase components and the synthesis bank reconstructs the input from the polyphase components.

It is easy to express the vector of analysis filters in the following form:

$$\begin{bmatrix} H_0(\mathbf{z}) \\ H_1(\mathbf{z}) \\ \\ H_{M-1}(\mathbf{z}) \end{bmatrix} = \mathbf{H}_P\left(\mathbf{z}^{\mathbf{D}}\right)\mathbf{z}^{-\mathbf{M}} \quad (26)$$

where $\mathbf{M}$ is the matrix whose lth column is polyphase shift vectors $\mathbf{m}_l$. Therefore, the lth column of $\mathbf{H}_{AC}(\mathbf{z})$, which is simply the vector $\mathbf{z}$ of analysis filters aliased by the shift vector $e_{\mathbf{D}^{-1}}(2\pi\mathbf{k}_l)$ of cosets for l=0,1,…,M-1, is given by

$$\begin{bmatrix} H_{0l}(\mathbf{z}) \\ H_{1l}(\mathbf{z}) \\ \vdots \\ H_{(M-1)l}(\mathbf{z}) \end{bmatrix} = \begin{bmatrix} H_0\left(e_{\mathbf{D}^{-1}}(2\pi\mathbf{k}_l)\mathbf{z}\right) \\ H_1\left(e_{\mathbf{D}^{-1}}(2\pi\mathbf{k}_l)\mathbf{z}\right) \\ \vdots \\ H_{M-1}\left(e_{\mathbf{D}^{-1}}(2\pi\mathbf{k}_l)\mathbf{z}\right) \end{bmatrix}$$

$$= \mathbf{H}_P\left(\left[e_{\mathbf{D}^{-1}}(2\pi\mathbf{k}_l)\mathbf{z}\right]^D\right)\left[e_{\mathbf{D}^{-1}}(2\pi\mathbf{k}_l)\mathbf{z}\right]^{-\mathbf{M}}$$

$$= \mathbf{H}_P\left(\mathbf{z}^{\mathbf{D}}\right)\left[e_{\mathbf{D}^{-1}}(2\pi\mathbf{k}_l)\mathbf{z}\right]^{-\mathbf{M}} \quad (27)$$

Putting the M columns of $\mathbf{H}_{AC}(z)$ together, we have

$$\mathbf{H}_{AC}(\mathbf{z}) = \mathbf{H}_P\left(\mathbf{z}^{\mathbf{D}}\right) \mathrm{diag}\left(\mathbf{z}^{-\mathbf{m}_0}, \cdots, \mathbf{z}^{-\mathbf{m}_{M-1}}\right)\mathbf{W} \quad (28)$$

where $\mathbf{W}$ is a matrix with the (n,i)th entry element $e^{i2\pi \mathbf{k}_i^T \mathbf{m}_n}$. It is also possible to write the row vector of synthesis filters

$$\left[G_0(\mathbf{z}) \quad G_1(\mathbf{z}) \quad \cdots \quad G_{M-1}(\mathbf{z})\right] = \left(\mathbf{z}^{\mathbf{m}}\right)^T \mathbf{G}_p\left(\mathbf{z}^{\mathbf{D}}\right) \quad (29)$$

Inserting (28) and (29) into (21), we have

$$Y(\mathbf{z}) = \frac{1}{M}\left[G_0(\mathbf{z}), \cdots, G_{M-1}(\mathbf{z})\right] = \mathrm{H}_P\left(\mathbf{z}^{\mathbf{D}}\right)\mathrm{diag}\left(\mathbf{z}^{-\mathbf{m}_0}, \cdots, \mathbf{z}^{-\mathbf{m}_{M-1}}\right)$$
$$\mathbf{W}\begin{bmatrix} X\left[e_{\mathbf{D}^{-1}}(2\pi\,\mathbf{k}_0)\mathbf{z}\right] \\ X\left[e_{\mathbf{D}^{-1}}(2\pi\,\mathbf{k}_1)\mathbf{z}\right] \\ \vdots \\ X\left[e_{\mathbf{D}^{-1}}(2\pi\,\mathbf{k}_{M-1})\mathbf{z}\right] \end{bmatrix} \quad (30)$$

2. Reversal in a Two-Dimensional Two-Channel Filter Bank

We will consider a two-dimensional two-channel PRFB based on the decimation matrix:

$$\mathbf{D} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \quad (31)$$

and the polyphase shift vectors:

$$\mathbf{m}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ and } \mathbf{m}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad (32)$$

The polyphase form of such a system is given by

$$\begin{bmatrix} H_0(\mathbf{z}) \\ H_1(\mathbf{z}) \end{bmatrix} = \mathbf{H}_P\left(\mathbf{z}^{\mathbf{D}}\right)\mathbf{z}^{-\mathbf{M}} = \left[\mathbf{H}_P\left(\mathbf{z}^{\mathbf{D}}\right)\right]\begin{bmatrix} \mathbf{z}^{-\mathbf{m}_0} \\ \mathbf{z}^{-\mathbf{m}_1} \end{bmatrix} \quad (33)$$

Where $\mathbf{z} = [z_1 \quad z_2]$, $\mathbf{z}^{-\mathbf{m}_0} = z_1^{-m_{01}} z_2^{-m_{02}} = 1$, and $\mathbf{z}^{-\mathbf{m}_1}$ =

$z_1^{-m_{11}} z_2^{-m_{12}} = z_1^{-1}$.

Hence the analysis filters z-transform in the form of the analysis polyphase matrix as:

$$\begin{cases} H_0(\mathbf{z}) = \mathbf{H}_p(\mathbf{z}^{\mathbf{D}}) \\ H_1(\mathbf{z}) = \mathbf{H}_P(\mathbf{z}^{\mathbf{D}}) z_1^{-1} \end{cases} \quad (34)$$

Similar to the synthesis filters

$$[G_0(\mathbf{z}) \quad G_1(\mathbf{z})] = (\mathbf{z}^{\mathbf{m}})^T \mathbf{G}_p(\mathbf{z}^{\mathbf{D}}) = \begin{bmatrix} 1 \\ z_1 \end{bmatrix}^T \mathbf{G}_P(\mathbf{z}^{\mathbf{D}}) \quad (35)$$

or in the column structure, written as

$$\begin{cases} G_0(\mathbf{z}) = \mathbf{G}_P^T(\mathbf{z}^{\mathbf{D}}) \\ G_1(\mathbf{z}) = \mathbf{G}_P^T(\mathbf{z}^{\mathbf{D}}) z_1 \end{cases} \quad (36)$$

We shall impose space reversal as well as modulation on the analysis bank. In the discrete-space domain, the space-reversal of a filter $h(\mathbf{n})$ is accomplished by replacing $\mathbf{n}$ by$-\mathbf{n}$, and the effect in the z-domain is replaced $\mathbf{z}$ by $\mathbf{z}^{-\mathbf{I}}$. Let $H_1(\mathbf{z})$ be related to $H_0(\mathbf{z})$ by modulation followed by space reversal with an additional shift $\mathbf{z}^p$, that is

$$H_1(\mathbf{z}) = \mathbf{z}^p H_0((-\mathbf{z})^{-\mathbf{I}}) = \mathbf{z}^p H_0(-\mathbf{z}^{-\mathbf{I}}) \quad (37)$$

Since $H_0(\mathbf{z})$ can be written in the polyphase form as

$$H_0(\mathbf{z}) = H_{00}(\mathbf{z}^{\mathbf{D}}) + H_{01}(\mathbf{z}^{\mathbf{D}}) z^{-\mathbf{m}_1} \quad (38)$$

$H_1(\mathbf{z})$ can be rewritten as

$$H_1(\mathbf{z}) = \mathbf{z}^p \left[H_{00}(\mathbf{z}^{-\mathbf{D}}) - H_{01}(\mathbf{z}^{-\mathbf{D}})(\mathbf{z}^{\mathbf{m}_1}) \right] \quad (39)$$

If we select p=$-\mathbf{m}_1$, then the equation above can be written as

$$H_1(\mathbf{z}) = \mathbf{z}^{-\mathbf{m}_1} H_{00}(\mathbf{z}^{-\mathbf{D}}) - H_{01}(\mathbf{z}^{-\mathbf{D}}) \quad (40)$$

This expression nothing more than $H_1(\mathbf{z})$ in terms of its polyphase components, which are therefore related to those of $H_0(\mathbf{z})$ by

$$\begin{cases} H_{10}(\mathbf{z}) = -H_{01}(\mathbf{z}^{-\mathbf{I}}) \\ H_{11}(\mathbf{z}) = H_{00}(\mathbf{z}^{-\mathbf{I}}) \end{cases} \quad (41)$$

Thus the symmetry imposed by modulation and space reversal cause the analysis polyphase matrix to take the form

$$\mathbf{H_p}(\mathbf{z}) = \begin{bmatrix} H_{00}(\mathbf{z}) & H_{01}(\mathbf{z}) \\ H_{10}(\mathbf{z}) & H_{11}(\mathbf{z}) \end{bmatrix} = \begin{bmatrix} H_{00}(\mathbf{z}) & H_{01}(\mathbf{z}) \\ -H_{01}(\mathbf{z}^{-\mathbf{I}}) & H_{00}(\mathbf{z}^{-\mathbf{I}}) \end{bmatrix} \quad (42)$$

The analysis/synthesis (AS) symmetry relation can be enforced by choosing each synthesis filter to be a space reversal version of the corresponding analysis filter; that is, we assume that

$$\begin{cases} G_0(\mathbf{z}) = H_0(\mathbf{z}^{-\mathbf{I}}) = H_{00}(\mathbf{z}^{-\mathbf{D}}) + H_{01}(\mathbf{z}^{-\mathbf{D}}) z^{\mathbf{m}_1} \\ G_1(\mathbf{z}) = H_1(\mathbf{z}^{-\mathbf{I}}) = -H_{01}(\mathbf{z}^{\mathbf{D}}) + H_{00}(\mathbf{z}^{\mathbf{D}}) z^{\mathbf{m}_1} \end{cases} \quad (43)$$

$$\mathbf{G_p}(\mathbf{z}) = \begin{bmatrix} G_{00}(\mathbf{z}) & G_{10}(\mathbf{z}) \\ G_{01}(\mathbf{z}) & G_{11}(\mathbf{z}) \end{bmatrix} = \begin{bmatrix} H_{00}(\mathbf{z}^{-\mathbf{I}}) & -H_{01}(\mathbf{z}) \\ H_{01}(\mathbf{z}^{-\mathbf{I}}) & H_{00}(\mathbf{z}) \end{bmatrix} \quad (44)$$

Thus, the condition for PR hence becomes

$$\begin{bmatrix} H_{00}(\mathbf{z}^{-\mathbf{I}}) & -H_{01}(\mathbf{z}) \\ H_{01}(\mathbf{z}^{-\mathbf{I}}) & H_{00}(\mathbf{z}) \end{bmatrix} \begin{bmatrix} H_{00}(\mathbf{z}) & H_{01}(\mathbf{z}) \\ -H_{01}(\mathbf{z}^{-\mathbf{I}}) & H_{00}(\mathbf{z}^{-\mathbf{I}}) \end{bmatrix} = \mathbf{I} \quad (45)$$

$$\begin{bmatrix} H_{00}(\mathbf{z}^{-\mathbf{I}}) H_{00}(\mathbf{z}) + H_{01}(\mathbf{z}) H_{01}(\mathbf{z}^{-\mathbf{I}}) & 0 \\ 0 & H_{01}(\mathbf{z}^{-\mathbf{I}}) H_{01}(\mathbf{z}) + H_{00}(\mathbf{z}) H_{00}(\mathbf{z}^{-\mathbf{I}}) \end{bmatrix} = \mathbf{I} \quad (46)$$

IV. CONCLUSION

In this paper, we present a filter bank to achieve PR, and it is necessary and sufficient that the aliasing component matrix be invertible and the synthesis filters be chosen as the first column of $\mathbf{H}_{AC}^{-T}(\mathbf{z})$, under the assumptions reversal in the space domain by $\mathbf{z}$ to $-\mathbf{z}$, and shift by a factor $\mathbf{z}^{-\mathbf{M}}$, the three terms of analysis components $H_1(\mathbf{z})$ and synthesis components $G_0(\mathbf{z})$, $G_1(\mathbf{z})$ can be expressed in the form of $H_{00}(\mathbf{z})$ and $H_{01}(\mathbf{z})$. So it takes only in the diagonal constraint to be explicitly enforced to let it be the monomial 1, which reduces the procedure that we need the number of variables and equations at the beginning.

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