Journal of Science and Engineering Technology, Vol. 15, No. 2, pp. 37~43 (2019)

A New Numerical Method and Its Application in Digital Systems

CHUEN-MING CHEN* and PIN-HAN LI
Department of Computer Science & Information Engineering
Taipei City University of Science and Technology
No. 2, Xueyuan Rd., Beitou, Taipei 11201, Taiwan, R. O. C.
******cm.chen@tpcu.edu.tw*

ABSTRACT

This paper proposes a newly derived method to improve the traditional integral method. The trapezoidal integration method (also called the bilinear transform) compensates an improved integral area; therefore, the new numerical integral method is more accurate than the trapezoidal integration method. This new technique can be used to convert an original continuous-time system into an equivalent discrete-time system more precisely. In its application in a digital signal system, the newly generated digital transform has better accuracy than the bilinear transform does. An analysis of the mathematical transformation is given in this paper to show its adjustable characteristics for digital signal systems. An illustrated example was simulated on a real system using the developed method to demonstrate its time-domain response.

Key Words**:** numerical integral, digital system, bilinear transform

一種新型數值方法與其在數位系統之應用

陳春明*　李品翰
臺北城市科技大學資訊工程系
11201 台北市北投區學園路 2 號
******cm.chen@tpcu.edu.tw*

摘　要

本文中，提出一種新型態的推導方法，以改進傳統的積分方法。針對梯型積分法(又稱為雙線性轉換)，用補償的方法改善了積分面積量，使得新產生的數值積分比梯型積分法更加精確。利用此一新方法，可以更精準地將原始連續時間系統換成對等的數位時間數位系統。在數位化信號系統的應用上，新產生的數位轉換比雙線性轉換呈現更佳的精確性。文中的數學轉換分析顯示了對數位信號的可調特性。舉出一個實際系統並用文中所提出的技巧模擬，展現在時域響應中。

關鍵詞：數值積分，數位系統，雙線性轉換

I. INTRODUCTION

In the analysis of a signal system, the system equation in continuous-time domain is the differential equation, and the s transformation method (Laplace transformation method) is its concerning description. On the other hand, the system equation in discrete-time domain is the difference equation, and the z transform method is used for the analysis or design of the digital system. There are several methods [4, 7] commonly available for obtaining discrete-time equivalents of continuous-time signal filters (or analog filters). A discrete-time equivalent of a continuous-time filter must have approximately the same dynamic characteristics as the original continuous-time filter. In other words, in obtaining a discrete-time equivalent of a continuous-time filter, it is desired to have transient and steady state responses. The relationship between these two domain transform can be described and represented by using many existing numerical method [2, 3]. Especially, the famous trapezoidal integration, it is also well known as the bilinear transformation method or Tustin transformation method. The useful numerical integration between continuous-time and discrete-time responses can be conveniently applied in the analysis of analog and digital systems. However, the bilinear transformation method still can be further improved by the use of a proposed compensation technique.

Due to the high progress in microprocessor and digital computer, sampled-data system could preserve high reliability and low electronic noise. Usually, a digital controller is converted from its analog version because that many well-developed techniques are so familiar in the continuous-time domain. For example, PID designed method, phase compensated method, and some developed method [8] always perform on the continuous-time model. Thus, how to digitalize an analog controller to an equivalent counterpart is very attractive. It is known that there are various discretization methods via approximate transformation [1, 5, 6]. All the different methods have the same purpose to redesign the original system precisely, then the state response of sampled-data system could accurately match that of continuous system with the same inputs and initial condition. In this paper, another transformation is proposed to obtain a newly equivalently digital representation of an analog controller. By utilizing a compensated approach, a superior digital

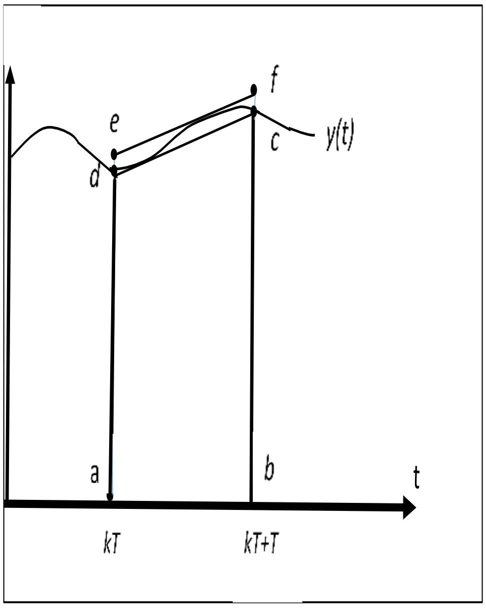

Fig. 1. The bilinear integration and a compensated area

transformation is easily to be accomplished. Base on the proposed method, an example is demonstrated to show the effectiveness.

II. NUMERICAL INTEGRATION

Let us consider the signal response as shown in Fig. 1. The response $y(t)$ is in sampling by a sampling period T as indicated in the Fig. 1. The trapezoidal area A_1 (area abcd) is evaluated as

$$A_1 = \frac{1}{2}(upper\ base + bottom\ base) * height \tag{1}$$

The upper base is *y(kT)*, the bottom base is *y(kT+T)*, and the height is T, thus the trapezoidal area A_1 is

$$A_1 = \frac{1}{2}(y(kT) + y(kT+T)) \tag{2}$$

It is seen that the continuous-time integral area between $kT \sim kT+T$ can be improved by a compensated area A_2 (area dcfe) The A_2 is a parallelogram area given as

$$A_2 = bottom * height \tag{3}$$

The height is T, and the bottom (the length of $\overline{cf}$ or $\overline{de}$) is using the fractional difference between *y(kT)* and *y(kT+T)*, thus the parallelogram area A_2 is

$$A_2 = \frac{1}{n}\big(y(kT+T) - y(kT)\big)T \tag{4}$$

where the fractional number n is adjustable. From the viewpoint of geometric curve, the integration result of $A_1 + A_2$ is more accurate than A_1. It is believed that the trapezoidal area plus compensated parallelogram area can improve the response integration. Using such an approach, the discrete-time transformation concerning a signal system derived from the analog response will have a better performance.

III. THE APPLICATION ON DIGITAL SYSTEMS

A continuous-time filter described by the transfer function is

$$\frac{Y(s)}{X(s)} = \frac{a}{s+a} \tag{5}$$

and its differential equation is

$$\frac{d\,y(t)}{d\,t} + a\,y(t) = a\,x(t) \tag{6}$$

The sampling period *T* is considered to determine the value of y(t), eq.(6) is given by the difference equation

$$\begin{aligned} &y(kT+T) - y(kT) \\ &= -a\int_{kT}^{(k+1)T} y(t)\,dt + a\int_{kT}^{(k+1)T} x(t)\,dt \end{aligned} \tag{7}$$

There are several methods [2-4, 7] for obtaining discrete-time equivalents of continuous-time filters. From the viewpoint of geometric meaning, the calculation of each integral term on the right-hand side of (7) can be numerically integrated by the approach of trapezoidal area plus a compensated parallelogram area. As the description given in the forward section. the area of the compensated parallelogram is

$$T\,\frac{1}{n}\big[y(kT+T) - y(kT)\big] \tag{8}$$

where *n* is a chosen number. Based on such an approach, (7) can be written as

$$\begin{aligned} &y(kT+T) - y(kT) \\ &= -aT\left\{\frac{1}{2}\big[y(kT+T) + y(kT)\big] + \frac{1}{n}\big[y(kT+T) - y(kT)\big]\right\} \\ &\quad + aT\left\{\frac{1}{2}\big[x(kT+T) + x(kT)\big] + \frac{1}{n}\big[x(kT+T) - x(kT)\big]\right\} \end{aligned} \tag{9}$$

The z transform of (9) is

$$\begin{aligned} &zY(z) - Y(z) \\ &= -aT\{\frac{1}{2}[zY(z) + Y(z)] + \frac{1}{n}[zY(z) - Y(z)]\} \\ &\quad + aT\{\frac{1}{2}[zX(z) + X(z)] + \frac{1}{n}[zX(z) - X(z)]\} \end{aligned} \tag{10}$$

After rearrangement the above equation, it obtains

$$\begin{aligned} &\left[z(1 + \frac{aT}{2} + \frac{aT}{n}) - 1 + \frac{aT}{2} - \frac{aT}{n} \right] Y(z) \\ &= \left[z(\frac{aT}{2} + \frac{aT}{n}) + \frac{aT}{2} - \frac{aT}{n} \right] X(z) \end{aligned} \tag{11}$$

and the discrete-time transfer function yields

$$\frac{Y(z)}{X(z)} = \frac{a}{\dfrac{2n(z-1)}{T\big[(n+2)z + (n-2)\big]} + a} \tag{12}$$

By comparing (5) and (12), the relationship of *s* and *z* is

$$s = \frac{2n(z-1)}{T\big[(n+2)z + (n-2)\big]} \tag{13}$$

the above equation also can be rearranged as

$$s = \frac{2n(z-1)}{T\big[n(z+1) + 2(z-1))\big]} \tag{14}$$

In (14), it is worthy of noting that *n*→ ∞, (14) results in the bilinear transformation (since the compensated area in (4) is zero), it gives

$$s = \frac{2(z-1)}{T(z+1)} \tag{15}$$

It is seen that the bilinear transform is a special case as $n \to \infty$ in the derived results of our proposed method..

On the consideration of stability about the new transformation, it is interested to discuss the stability region, the left half of the s-plane (Re(s)<0) is mapped into the region

$$\mathrm{Re}\left(\frac{2}{T} \cdot \frac{n(z-1)}{[(n+2)z+(n-2)]}\right) < 0 \tag{16}$$

By substituting $z = \sigma + j\omega$, the above inequality becomes

$$\mathrm{Re}\left(\frac{n(\sigma + j\omega - 1)}{(n+2)(\sigma + j\omega) + (n-2)}\right) < 0 \tag{17}$$

or

$$(n\sigma - n)(n\sigma + n + 2\sigma - 2) + n\omega^2(n+2) < 0 \tag{18}$$

Rearranging (18) to obtain

$$\left(\sigma - \frac{2}{n+2}\right)^2 + \omega^2 < \left(\frac{n}{n+2}\right)^2 \tag{19}$$

as $n \to \infty$, it becomes the stability region within the unit circle (the original stability region). Therefore, it is obvious that our new transformation in (14) is a reasonable result. Moreover, it is seen that the transformation via the compensated method can map the left half of the *s* plane into a stable circle region of the *z* plane. When we choose different *n*, the relatively mapping stable regions of z plane are also different. Therefore, the proposed transform method can also produce a stable discrete-time filter for a stable continuous-time filter via a chosen *n*.

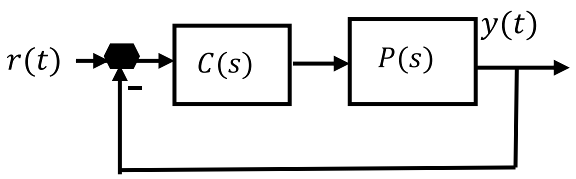

Fig.2. The analog signal system

IV. THE EXAMPLE ON SIGNAL SYSTEM

Let us consider the analog signal system with the designed compensator [1] as shown in Fig.2. Where *C*(s) and *P*(s) represent the filter (or controller) and plant respectively, the reference input $r(t)$ is an unit-step function,

$$C(s) = \frac{s^2 + 10.42s + 20}{s^2 + 32.44s + 20} \tag{20}$$

$$P(s) = \frac{20}{s(1+\frac{s}{10})(1+\frac{s}{30})} \tag{21}$$

The corresponding transfer function of controller from the bilinear transform rule (the result in (15)) is

$$C(z) = C(s)\Big|_{s=\frac{2}{T}\frac{z-1}{z+1}} \tag{22}$$

or

$$\left.\frac{s^2 + 10.42s + 20}{s^2 + 32.44s + 20}\right|_{s=\frac{2}{T}\frac{z-1}{z+1}} = \frac{num_{bilinear}}{den_{bilinear}} \tag{23}$$

where

$$\begin{aligned} num_{bilinear} = {} & [4 + 20.84T + 20T^2]z^2 \\ & + [-8 + 40T^2]\, z \\ & + [4 - 20.84T + 20T^2] \end{aligned} \tag{24}$$

$$\begin{aligned} den_{bilinear} = {} & [4 + 64.88T + 20T^2]z^2 \\ & + [-8 + 40T^2]z \\ & + [4 - 64.88T + 20T^2] \end{aligned} \tag{25}$$

The corresponding transfer function of controller from the transformation of our proposed method (the result in (13)) is

$$\left.\frac{s^2 + 10.42s + 20}{s^2 + 32.44s + 20}\right|_{s=\frac{2}{T}\frac{n(z-1)}{[(n+2)z+(n-2)]}} = \frac{num_{proposed}}{den_{proposed}} \tag{26}$$

where

$$num_{proposed} = [4n^2 + 20.84n(n+2)T + 20(n+2)^2T^2]z^2 + [-8n^2 - 83.36nT + 40(n^2-4)T^2]z + [4n^2 - 20.84n(n-2)T + 20T^2(n-2)^2] \quad (27)$$

$$den_{proposed} = [4n^2 + 64.88n(n+2)T + 20(n+2)^2T^2]z^2 + [-8n^2 - 259.52nT + 40(n^2-4)T^2]z + [4n^2 - 64.88n(n-2)T + 20T^2(n-2)^2] \quad (28)$$

The rule of discretization of analog plant is

$$P(z) = z.o.h\{P(s)\} = Z\{\frac{1-e^{-sT}}{s}P(s)\} \quad (29)$$

The equivalent discrete controlled system is shown in Fig. 3, and the output of the discrete controlled system is

$$y(kT) = Z^{-1}\left\{\frac{C(z)P(z)}{1+C(z)P(z)} \cdot R(z)\right\} \quad (30)$$

Substituting the bilinear transformation and our new proposed transformation into Fig. 3, we can obtain the two different discrete-time responses.

For comparing the performance between bilinear transform and our proposed transform method, sampling period T=0.05[sec] is utilized, and their output responses of two methods are shown in the following Fig. 4.

We choose the adjustable variable $n = 3$ in the discrete-time transform of this simulation responses in Fig.4. It can note that the new proposed transformation by our proposed method is more closely match the analog response. Moreover, for precisely comparison, let $E_{bilinear}$ is error sum of bilinear transform ($y_b(kT)$ be the bilinear response) and $E_{proposed}$ be the error sum of our proposed transform ($y_p(kT)$ is our

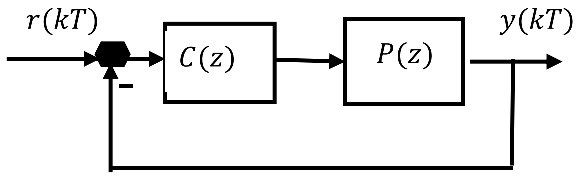

Fig. 3. The digital signal system

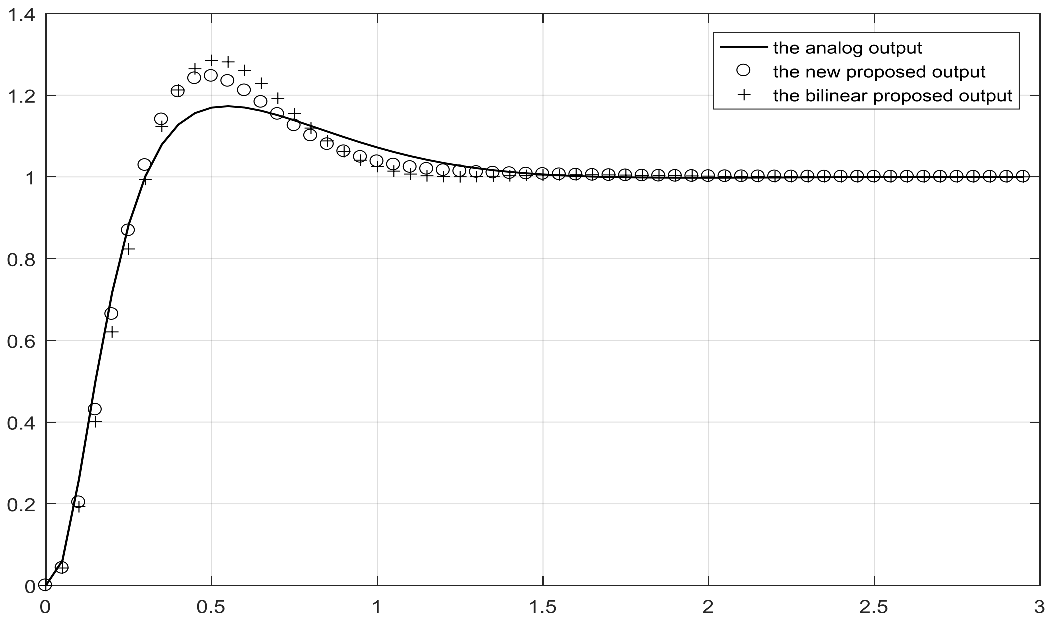

Fig.4. The comparison of output responses in analog system, bilinear digital system, and the proposed digital system

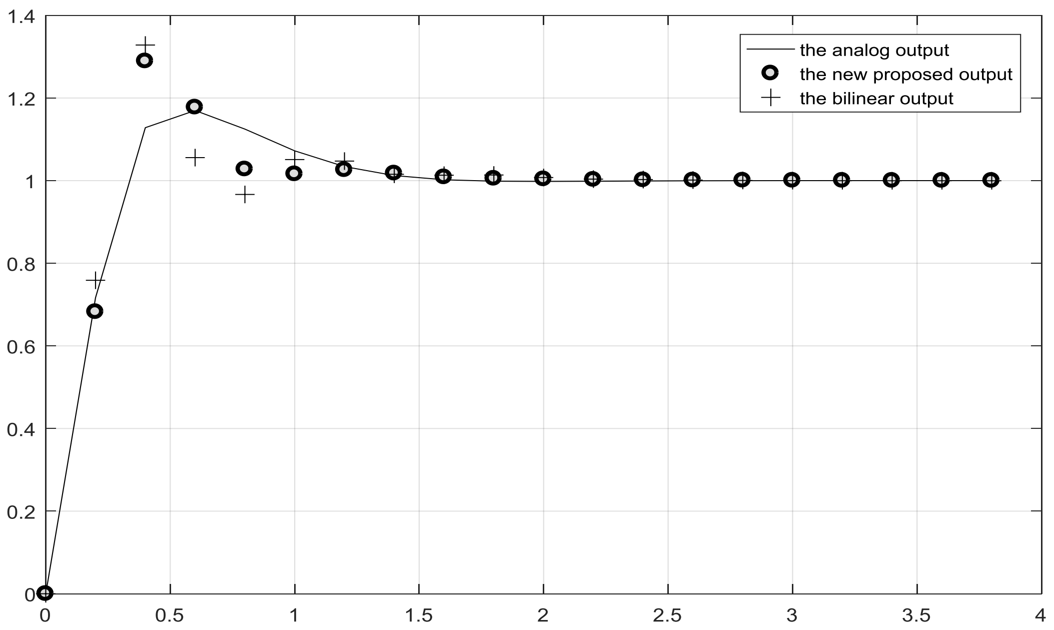

Fig.5. The comparison of output responses in analog system, bilinear digital system, and the proposed

proposed response) shown as

$$\boldsymbol{E_{bilinear}} = \sum_{k=1}^{k=60}[y(t = kT) - y_b(kT)]^2 = 0.1028 \qquad (31)$$

$$E_{proposed} = \sum_{k=1}^{k=60}[y(t = kT) - y_p(kT)]^2 = 0.0501 \qquad (32)$$

In order to test a different variable *n* and a larger sampling period T, we exaggeratedly choose $n = 4$ and the sampling period *T*=0.2[sec], and the output responses of two methods are shown in the following Fig.5., The error sum in Fig. 5 are given as

$$E_{bilinear} = \sum_{k=1}^{k=20}[y(t = kT) - y_b(kT)]^2 = 0.0809 \qquad (33)$$

$$E_{proposed} = \sum_{k=1}^{k=20}[y(t = kT) - y_p(kT)]^2 = 0.0401 \qquad (34)$$

V. CONCLUSIONS

A newly compensated method has been discussed to convert the analog filter or controller to its corresponding counterpart of digital transformation, and the application on the signal system. It is shown that the proposed method can more accurately match the continuous system than bilinear method by choosing an adjustable variable *n*. It is believed that the new techniques can be applied in the field of signal and control systems for systematic analysis and signal processing.

REFERENCES

1. Chen, C. F. and L. S. Shieh (1970) An algebraic method for control systems design. *International Journal of Control*, 11(5), 717-739.
2. Chen, C. T. (1993) Analog and Digital Control System Design: Transfer-function, State-space, and Algebraic Method. *Saunders college publishers, International edition*.
3. Gene, F., J. Franklin, D. Powell and M. L. Workman (1990) *Digital Control of Dynamic Systems*, Addison-Wesley Publishing Company, Inc.
4. Kuo, B. C. (1980) *Digital Control Systems*, Holt, Rinehart and Winston, New York.
5. Rosenvasser, Y. N., K. Y. Polyakov and B. P. Lampe (1999) Application of Laplace Transformation for Digital Redesign of Continuous Control Systems. *IEEE Transactions on Automatic Control*. 44(4), 883-886.
6. Sung, Y. G., W. S. Jang and J. Y. Kim (2018) Negative Input Shaped Commands for Unequal Acceleration and Braking Delays of Actuators. *Journal of Dynamics*

Systems, Measurement, and Control, 140, Paper No. 094501.

7. Veloni, A. and N. I. Miridakis (2018) *Digital Control Systems–Theoretical Problems and Simulation Tools,* CRC Press, Taylor & Franics Groug, New York.
8. Wamg, B., J. Cheng, A. Albarakati and H. M. Fardon, (2017) A Mismatched membership Function Approach to Sampled-data Stabilization for T-S Fuzzy Systems with time-varying delayed signals. *Signal Processing*, 140, 161-170.

收件：**108.04.10**　修正：**108.06.10**　接受：**108.07.17**