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On the Cardinality of Permutations for Interval Exchange Transformations

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ABSTRACT

The basic problem of *enumerative combinatorics* is counting the number of elements for a set. This paper focuses on a particular set G_N, which is the subset of permutations of N items for *interval exchange transformations*. In mathematics, an *interval exchange transformation* is a type of dynamical system. Unlike a "sieve" method that begins with a larger set and somehow eliminates the unqualified elements, a decomposition approach was used in this study. Based on the results of using this approach, we propose a concise formula of the cardinality of G_N. In addition, we related the set of $G_{N,N}$ to the set of $B_{N,N}$, where $G_{N,N}$ denotes the subset of G_N that is composed of all permutations with a prefix "N", and $B_{N,N}$ denotes the set of permutations without a *succession*. For $N \geq 1$, we proved and thus propose that $B_{N,N}$ and $G_{N+1,N+1}$ are *isomorphic* and that $B_{N,N}$ is *postequivalent* to $G_{N+1,N+1}$.

***Key Words*:** enumerative combinatorics, interval exchange transformation, permutation, derangements, isomorphic.

針對區段交換轉換之排列的計數

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摘　要

計數組合學的基本問題是研究如何計算一個特定的有限集合之組成個數。本研究選擇 n 項排列的一個特定的部份集合 G_N 作爲研究對象，此部份集合乃是提供運作於 n 個區段上的區段交換轉換之用。就數學上而言，區段交換轉換是一種動態系統。針對 G_N 的計數，本研究採用拆解法而有別於常見的篩選法。利用此方法我們針對 G_N 的個數提出一個很簡潔的計算公式。除此之外，我們將集合 $G_{N,N}$ 聯結至集合 $B_{N,N}$。此處，$G_{N,N}$ 是集合 G_N 的一個部份集合，它是以 N 起頭的所有排列；而 $B_{N,N}$ 是一個由沒有任何兩個相鄰遞增位置是相鄰遞增數字的所有排列組成的集合。我們提出並證明當 $N \geq 1$ 時 $G_{N+1,N+1}$ 與 $B_{N,N}$ 是同構的，而且 $B_{N,N}$ 是後-等同於 $G_{N+1,N+1}$。

關鍵詞：計數組合學，區段交換轉換，排列，錯位排列，同構。

I. Introduction

The basic problem of *enumerative combinatorics* is that of counting the number of elements of a set [6]. In this paper, we restrict attention to a particular set of permutations that arises naturally in the *interval exchange transformations*. In mathematics, an *interval exchange transformation*, first introduced by Katok and Stepin [2] and further studied by Keane [3], Veech [8] and many others, is a kind of dynamical system. An *interval exchange transformation* is obtained by cutting the unit interval into N subintervals according to a vector of N lengths, and then rearranging these subintervals according to a permutation of N items.

For convenience, we use notation S_N to denote the set of all permutations of N items $\{1,\ldots, N\}$. That is, a permutation $\boldsymbol{\pi} = (\pi_1\pi_2\ldots\pi_N)$ belongs to S_N if and only if

$$\pi_i \in \{1,\ldots,N\}, \quad \text{for all} \quad i=1,\ldots,N,$$

and

$$\pi_i \neq \pi_j, \quad \text{for all} \quad i \neq j.$$

Clearly, the cardinality of S_N is $N!$ (= $1\times2\cdots\times N$, called *N factorial*). However, it is not true that every permutation in S_N can be used in the *interval exchange transformations*. Actually, only a particular subset of S_N is qualified. This particular subset, denoted by G_N, should satisfy the following definition [7].

Definition 1. A permutation $\boldsymbol{\pi}$ belongs to G_N if an only if

$$\pi_{i+1} \neq \pi_i + 1, \quad \text{for all } 1 \leq i \leq N-1, \tag{1}$$

and

$$\sum_{j=1}^{i} \pi_j \neq \frac{1}{2} i(i+1), \quad \text{for all} \quad 1 \leq i \leq N-1. \tag{2}$$

By (2), it is clear that $\pi_1 \neq 1$ and that $\pi_N \neq N$.

Definition 2. Let g_N denote the cardinality of G_N.

It is also clear that $g_1 = 0$.

Little literature has been published on the topic of the cardinality of G_N. Only, to the best of my knowledge, Varol [7] presents a recursive formula as follows:

$$g_N = (N-1)(N-1)! - \sum_{i=2}^{N-1} g_i \left[\sum_{k=0}^{N-1} (N-i-k)! C_k^{i+k-1} \right] \tag{3}$$

The goal of this study is try to find a more concise formula than Varol's.

Varol's formula is derived by a "sieve" method which excluding those permutations that are not belonging to G_N from S_N [6]. Varol divides such permutations into $2N$-2 mutually disjoint subsets according to how they violate (1) and/or (2). By contrast, we adopt a decomposition approach to this problem. That is, we analyze the elements of G_N directly and decompose them into mutually exclusive subsets. By using this approach, in Section II, we propose several recursive formulas of the cardinality of G_N. In Section III, we relate the cardinality of G_N to a specific permutation problem. Conclusions are summarized in Section IV.

II. The cardinality of G_N

In this section we will propose several recursive formulas of the cardinality of G_N. The recursive formulas are in several different forms, depending on how we conduct the computation. First, in view of the elements of G_N, we give two definitions as follows.

Definition 3. Let $G_{i,N}$ denote the subset of G_N that is composed of all permutations with a prefix "*i*".

It is clear that $G_{1,N}$ is an empty set. Since, according to (2), permutations with a prefix "1" should not belong to G_N. Thus, for N = 4, we can decompose G_4 into three mutually exclusive subsets $G_{2,4}$, $G_{3,4}$, and $G_{4,4}$, such as those listed in Table 1.

Table 1. The Decomposition of G_4

	π_1	π_2	π_3	π_4
$G_{2,4}$	2	4	1	3
	2	4	3	1
$G_{3,4}$	3	1	4	2
	3	2	4	1
$G_{4,4}$	4	1	3	2
	4	2	1	3
	4	3	2	1

Once we have defined $G_{i,N}$, it is naturally to give the following definition.

Definition 4. Let $g_{i,N}$ denote the cardinality of $G_{i,N}$.

Thus, for $N = 4$, we have $g_{2,4} = 2$, $g_{3,4} = 2$, and $g_{4,4} = 3$, just as illustrated in Table 1.

Definitely, according to **Definitions** 2~4, we have

$$g_N = \sum_{i=2}^{N} g_{i,N}. \tag{4}$$

Table 2 presents some values of g_N and their corresponding values of $g_{i,N}$'s.

At first glance, the numbers in Table 2 satisfy identities such as

$$g_{3,N} = g_{2,N-1} + g_{2,N}, \tag{5}$$

And

$$g_{3,N} = g_{N-1} + g_{N-2}. \tag{6}$$

However, these identities are not good enough for us to compute g_N. Fortunately, by carefully observing the numbers in Table 2, it is not hard to come up with, for $N \geq 2$, the following recurrence relation.

Lemma 1.

$$g_{2,N} = g_{N-1} + (-1)^N, \tag{7}$$

Proof. The proof is by induction on *n*. First, (7) is trivially true for $N = 2$. Second, we assume that (7) is true for $N-1$; namely, we assume that $g_{2,N-1} = g_{N-2} + (-1)^{N-1}$. Now, we have to prove that $g_{2,N} = g_{N-1} + (-1)^N$. The idea is to try to write $g_{2,N}$ using $g_{2,N-1}$, which, by the induction hypothesis, is equal to $g_{N-2} + (-1)^{N-1}$. By (5) we have $g_{2,N} = g_{3,N} - g_{2,N-1} = g_{3,N} - (g_{N-2} + (-1)^{N-1})$. By (6) we have $g_{2,N} = g_{N-1} + g_{N-2} - (g_{N-2} + (-1)^{N-1})$

$$= g_{N-1} + (-1)^N. \blacksquare$$

Table 2. g_N vs. $g_{i,N}$'s

N \ $g_{i,N}$	$g_{2,N}$	$g_{3,N}$	$g_{4,N}$	$g_{5,N}$	$g_{6,N}$	$g_{7,N}$	g_N
1	0						**0**
2	**1**						**1**
3	**0**	**1**					**1**
4	**2**	2	**3**				**7**
5	**6**	8	8	**11**			**33**
6	**34**	40	42	42	**53**		**211**
7	**210**	244	250	256	256	**309**	**1525**

Beside, from Table 2, we conjectured the truth of the following recurrence relation:

$$g_{i,N} = g_{i-1,N} + g_{2,N-(i-2)} \times g_{i-1,i-1}. \tag{8}$$

This recurrence relation is supported by all the available data; but right now we have not yet been able to find a general proof. Fortunately, the predicted results of the conjecture are accordance with the results of Varol's [7].

Thus, by (4) we have

$$g_N = g_{N-1} + (-1)^N + \sum_{i=3}^{N} (g_{i-1,N} + g_{2,N-(i-2)} \times g_{i-1,i-1}). \tag{9}$$

These recurrence relations enable us to carry out a step-by-step computation of $g_{i,N}$'s and g_N. We call the computation a *left-to-right* computation since the order of computation, for $N = 5$, is $g_{2,5}$, $g_{3,5}$, $g_{4,5}$, and $g_{5,5}$ sequentially.

Alternatively, we also can conduct a *right-to-left* computation. That is, we first compute $g_{5,5}$, then compute $g_{4,5}$, $g_{3,5}$, and $g_{2,5}$ sequentially. For $N \geq 3$, since $g_{2,2} = 1$, the recurrence relation of (8) can be simplified to

$$g_{N,N} = g_{N-1,N} + g_{N-1,N-1}. \tag{10}$$

Besides, we have the following lemma.

Lemma 2.

$$g_{N,N} = \frac{!N}{N-1}. \tag{11}$$

Proof. According to the sequence A000255 in Sloane's On-Line Encyclopedia of Integer Sequences [4], the notation !N, read "N

subfactorial", denotes the number of *derangements* of N items which follows that

$$!N = \frac{N!}{0!} - \frac{N!}{1!} + \cdots + (-1)^N \frac{N!}{N!}. \qquad (12)$$

Derangements of $\{1, 2, \ldots, N\}$ are those permutations $\boldsymbol{\pi} = (\pi_1\pi_2\ldots\pi_N)$ such that $\pi_1 \neq 1$, $\pi_2 \neq 2, \ldots$, and $\pi_N \neq N$. For example, when N = 2, 3, 4, 5, 6, and 7, the values of $!N$ are 1, 2, 9, 44, 265, and 1854, respectively. Hence, the values of $!N/(N\text{-}1)$ are 1, 1, 3, 11, 53, and 309, respectively. These numbers are exactly those $g_{N,N}$'s in the diagonal of Table 2. ■

Besides, by carefully observing the diagonal of Table 2, we find another recurrence relation, for $N \geq 3$, as the following lemma.

Lemma 3.

$$g_{N,N} = (N-2) \times g_{N-1,N-1} + (N-3) \times g_{N-2,N-2}. \qquad (13)$$

Proof. If we can prove that the right-hand side of (13) is equal to the right-hand side of (11), then we can prove that (13) is valid. Now, applying (11), we can rewrite the right-hand side of (13) as follows:

$$!(N-1)+!(N-2). \qquad (14)$$

Hence, according to (12), we can rewrite (14) as follows:

$$\frac{N(N-2)!}{0!} - \frac{N(N-2)!}{1!} + \cdots + (-1)^{N-2}\frac{N(N-2)!}{(N-2)!} + (-1)^{N-1}\frac{(N-1)!}{(N-1)!}. \qquad (15)$$

Similarly, according to (12), we can rewrite the right-hand side of (11) as follows:

$$\frac{N(N-2)!}{0!} - \frac{N(N-2)!}{1!} + \cdots + (-1)^{N-1}\frac{N(N-2)!}{(N-1)!} + (-1)^{N}\frac{N(N-2)!}{N!}. \qquad (16)$$

Now, the first N-1 terms of both (15) and (16) are equivalent to each other. Besides, the last term of (15) is equivalent to the sum of the last two terms of (16). So, the right-hand side of (13) is equal to the right-hand side of (11). ■

Finally, we propose the following theorem.

Theorem 1.

$$g_N = \sum_{i=1}^{N-1} (i \times (g_i + (-1)^{i+1}) \times g_{N-i,N-i}). \qquad (17)$$

Proof. Once we have proved that (13) is valid, thus according to (10) and (13) we obtain the following recurrence relation

$$g_{N-1,N} = (N-3) \times (g_{N-1,N-1} + g_{N-2,N-2}). \qquad (18)$$

Besides, according to (7) and (8), we have the following identity

$$g_{N-2,N} = g_{N-1,N}. \qquad (19)$$

On the other hand, according to (8), we have

$$g_{i-1,N} = g_{i,N} - g_{2,N-(i-2)} \times g_{i-1,i-1}. \qquad (20)$$

Therefore, according to (13), (18), (19), and (20), by given $g_{N-1,N-1}$ and $g_{N-2,N-2}$, we can perform *right-to-left* computation for computing $g_{N,N}$, $g_{N-1,N}$, $g_{N-2,N}$ and so on sequentially.

However, neither *left-to-right* computation nor *right-to-left* computation is efficient. In fact, for computation of g_N we do not need to compute all $g_{i,N}$'s. Actually, by repeatedly applying (4), (7), and (8), we have another recursive formula, for $N \geq 2$, as follows:

$$g_N = \sum_{i=1}^{N-1} (i \times g_{2,i+1} \times g_{N-i,N-i}). \qquad (21)$$

Here, for the purpose of elegant expression, we embed a pseudo term $g_{1,1}$ that is set to one into the formula. Finally, by using (7), we can rewrite (21) as following

$$g_N = \sum_{i=1}^{N-1} (i \times (g_i + (-1)^{i+1}) \times g_{N-i,N-i}). \quad ■$$

Note that, by Definition 1, g_1 = 0. So, we can quickly compute g_N only by the diagonal of Table 2. Furthermore, by (11), the diagonal of Table 2 are already available. Here, for $2 \leq N \leq 5$, we list computations of g_N in Table 3.

Table 3. Computations of g_N, for $2 \le N \le 5$

N	g_N
2	$=1\times(g_1+1)\times1=1$
3	$=1\times(g_1+1)\times1+2\times(g_2-1)\times1=1$
4	$=1\times(g_1+1)\times1+2\times(g_2-1)\times1+3\times(g_3+1)\times1=7$
5	$=1\times(g_1+1)\times3+2\times(g_2-1)\times1+3\times(g_3+1)\times1+4\times(g_4-1)\times1=33$

So far, we have derived a bunch of recursive formulas of g_N. In the foregoing discussion, we know that $g_{N,N}$'s play an key role in the computation of g_N. Therefore, in Section III, we will further discuss $g_{N,N}$'s by relating it to a specific permutation problem.

III. $G_{N,N}$ vs. $B_{N,N}$

As a matter of fact, there is a compelling relationship between the set of $G_{N,N}$ and a set of permutations without a *succession*. By given a permutation in S_N, a maximal sequence of consecutive integers that appear in consecutive positions is called a *block* [4]. For example, in S_9, the permutation $\boldsymbol{\pi}$ = (456723189) contains four blocks namely 4567, 23, 1, and 89. In the context of *block*, we define the following two terms.

Definition 5. Let $B_{M,N}$ denote the subset of S_N that is composed of all permutations with *M* blocks.

Definition 6. Let $b_{M,N}$ denote the cardinality of $B_{M,N}$.

For example in S_3, it's easy to see that only one permutation, $\boldsymbol{\pi}$ = (123), contains one block namely 123; and that there are two permutations, $\boldsymbol{\pi}$ = (312) and $\boldsymbol{\pi}$ = (231), contain two blocks namely 3, 12, and 23, 1, respectively; and that the other three permutations, $\boldsymbol{\pi}$ = (132), $\boldsymbol{\pi}$ = (213), and $\boldsymbol{\pi}$ = (321), all contain three blocks namely 1, 2, and 3. However, the following question is not so easy to answer.

How many permutations in S_N contain exactly M blocks?

Fortunately, the answer has been given by Myers [4] as follows:

$$b_{M,N} = C_{M-1}^{N-1}(M-1)!\sum_{k=0}^{M-1}(-1)^{M-k-1}\frac{k+1}{(M-k-1)!}. \quad (22)$$

According to **Definition 6**, $b_{N,N}$ is the cardinality of $B_{N,N}$ that contains all permutations with no two consecutive increasing integers located in two consecutive positions. That is, permutations without a *succession*. For example, $b_{4,4}$ = 11, Table 4 shows the composition of $B_{4,4}$.

Table 4. The composition of $B_{4,4}$

π_1	π_2	π_3	π_4
1	3	2	4
1	4	3	2
2	1	4	3
2	4	1	3
2	4	3	1
3	1	4	2
3	2	1	4
3	2	4	1
4	1	3	2
4	2	1	3
4	3	2	1

Here, we introduce three definitions for helping to explain a theorem proposed in a few lines later [1].

Definition 7. A *combinatorial class*, or simply a class, is a finite or denumerable set on which a size function is defined, satisfying the following conditions:

i. The size of an element is a nonnegative integer;
ii. The number of elements of any given size is finite.

Definition 8. The *counting sequence* of a combinatorial class ***A*** is the sequence of integers $(A_n)_{n\ge0}$ where A_n is the number of elements in class ***A*** that have size *n*.

Definition 9. Two combinatorial classes ***A*** and ***B*** are *isomorphic*, if and only if their counting sequences are identical.

It is clear that both $B_{N,N}$ and $G_{N+1,N+1}$ are *combinatorial class*. Next, we propose and prove the following theorem.

Theorem 2. $B_{N,N}$ and $G_{N+1,N+1}$ are *isomorphic*. That is, $b_{N,N} = g_{N+1,N+1}$, for $N \ge 1$.

Proof[1]. In the foregoing discussion, we know that $b_{N,N}$ is the cardinality of $B_{N,N}$ that contains all permutations with no two consecutive increasing integers located in two consecutive positions. In other words, by definition, $B_{N,N}$ contains all permutations that satisfy (1). Let S'_{N+1} denote the subset of S_{N+1} that contain permutations in which $\pi_{N+1} = N+1$. Note that the cardinality of S_{N+1} is $(N+1)!$, and the cardinality of S'_{N+1} is $N!$. It is clearly that, in S'_{N+1}, there are $b_{N,N}$ permutations that their $\pi_1\pi_2...\pi_N$ satisfy (1). Now,

[1] Mathematical proof can be found in appendix.

move π_{N+1}, that is N+1, to front of these permutations, then these $b_{N,N}$ permutations satisfy (1) and (2), and we obtain the subset $G_{N+1,N+1}$. ■

Besides, we propose the following two definitions for helping to explain another theorem proposed in a few lines later.

Definition 10. Two combinatorial classes ***A*** and ***B*** are said to be *equivalent*, if and only if they satisfying the following conditions:

i. They are *isomorphic*;
ii. For every element a in ***A*** there is exactly one element b in ***B*** such that a is equal to b.

Definition 11. A combinatorial class ***A*** is said to be *post-equivalent* to a combinatorial class ***B***, if and only if they satisfying the following conditions:

i. They are *isomorphic*;
ii. For every element a in ***A*** there is exactly one element b in ***B*** such that a is equal to b, except for b's prefix part.

In this perspective, we propose and prove the following theorem.

Theorem 3. $B_{N,N}$ is *post-equivalent* to $G_{N+1,N+1}$, for $N \geq 1$.

Proof. By Theorem 2, we know that $B_{N,N}$ and $G_{N+1,N+1}$ are *isomorphic*. So, the first condition of Definition 11 is satisfied. Now, we shall consider the second condition of Definition 11. By Definition 1, we know that equation (1) means permutations without a *succession*. Since $B_{N,N}$ contains all permutations without a *succession*, they all satisfy (1). However, some of them violate (2). For example, N=4, there are four such permutations out of Table 4, namely $\boldsymbol{\pi}$ = (1324), $\boldsymbol{\pi}$ = (1432), $\boldsymbol{\pi}$ = (2143), and $\boldsymbol{\pi}$ = (3214), that satisfy (1) but violate (2). Clearly, if we prefixing every permutations of $B_{N,N}$ by an item N+1, then they not only satisfy (1) but also satisfy (2), and we obtain $G_{N+1,N+1}$. For example, if we prefixing an item "5" in front of every permutations of $B_{4,4}$ (as listed in Table 4), then we obtain $G_{5,5}$. ■

IV. Conclusions

We propose a decomposition approach to the problem of the cardinality of permutations which arises in the *interval exchange transformations*. The beauty of this approach lies not in the result itself, but rather in its wide applicability. We propose a concise formula of the cardinality of G_N that is simpler than those proposed by Varol [7]. Besides, we relate the set of $G_{N,N}$ to the set of $B_{N,N}$. We propose and prove that $B_{N,N}$ and $G_{N+1,N+1}$ are *isomorphic*, and that $B_{N,N}$ is *post-equivalent* to $G_{N+1,N+1}$, for $N \geq 1$.

V. Appendix

Proof. In the foregoing discussion, we know that the right-hand side of (11) can be recast as (16) and that the sum of the last two terms of (16) is equivalent to the last term of (15). Thus, if we let Y denote the last term of (15), then we can rewrite (16) as follows:

$$g_{N,N} = (N-2)!\left[\frac{N}{0!} - \frac{N}{1!} + \frac{N}{2!} + \cdots + (-1)^{N-2}\frac{N}{(N-2)!}\right] + Y.$$

Furthermore, it can be recast as

$$g_{N,N} = (N-2)!\left[\left(\frac{N}{0!} - \frac{1}{0!}\right) - \left(\frac{N}{1!} - \frac{2}{1!}\right) + \cdots + (-1)^{N-2}\left(\frac{N}{(N-2)!} - \frac{N-1}{(N-2)!}\right) + X\right] + Y.$$

Here,

$$X = \left[\frac{1}{0!} - \frac{2}{1!} + \frac{3}{2!} + \cdots + (-1)^{N-2}\frac{N-1}{(N-2)!}\right].$$

It is note that

$$\sum_{i=0}^{k}(-1)^i\frac{i+1}{i!} = \frac{(-1)^k}{k!}.$$

So, we have

$$X = \left[\frac{(-1)^{N-2}}{(N-2)!}\right].$$

Hence,

$$X(N-2)! = (-1)^{N-2}.$$

By recalling the last term of (15), we know that

$$Y=(-1)^{N-1}.$$

Thus,

$$X(N-2)!+Y=0.$$

Therefore, we obtain

$$g_{N,N}=(N-2)!\left[\frac{N-1}{0!}-\frac{N-2}{1!}+\frac{N-3}{2!}+\cdots+(-1)^{N-2}\frac{1}{(N-2)!}\right].$$

That is,

$$g_{N,N}=(N-2)!\sum_{k=0}^{N-2}(-1)^k\frac{N-k-1}{k!}.$$

On the other hand, by (22), we obtain

$$b_{N,N}=(N-1)!\sum_{k=0}^{N-1}(-1)^{N-k-1}\frac{k+1}{(N-k-1)!}$$

$$=(N-1)!\sum_{k=0}^{N-1}(-1)^k\frac{N-k}{k!}. \quad \blacksquare$$

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